

# Ooguri-Vafa Invariants and Off-shell Superpotentials of Type II/F-theory compactification

Feng-Jun Xu, Fu-Zhong Yang\*

*College of Physical Sciences, Graduate University of Chinese Academy of Sciences  
YuQuan Road 19A, Beijing 100049, China*

## Abstract

In this paper, we make a further step of [1] and calculate off-shell superpotential of two Calabi-Yau manifold with three parameters by integrating the period of subsystem. We also obtain the Ooguri-Vafa invariants with open mirror symmetry.

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\*Corresponding author E-mail: fzyang@gucas.ac.cn

# 1 Introduction

When Type II string theory compactfying on Calabi-Yau threefold with D-brane and background flux, the superpotential will be generated which in general can divided two parts—one originated from D-brane and the other from flux. The superpotential also plays an important role in mathematics which generates the Ooguri-Vafa invariants and counting the number of disks and sphere instantons.

For D5-brane wrapped the whole Calabi-Yau threefold, the holomorphic Chern-Simons theory [2]

$$\mathcal{W} = \int_X \Omega^{3,0} \wedge \text{Tr}[A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A] \quad (1.1)$$

gives the brane superpotential  $\mathcal{W}_{brane}$ , where  $A$  is the gauge field with gauge group  $U(N)$  for  $N$  D5-branes. When reduced dimensionally, the low-dimenaional brane superpotentials can be obtained as [3, 4]

$$\mathcal{W}_{brane} = N_\nu \int_{\Gamma^\nu} \Omega^{3,0}(z, \hat{z}) = \sum_\nu N_\nu \Pi^\nu \quad (1.2)$$

where  $\Gamma^\nu$  is a special Lagrangian 3-chain and  $(z, \hat{z})$  are closed-string complex structure moduli and D-brane moduli from open-string sector, respectively.

The background fluxes  $H^{(3)} = H_{RR}^{(3)} + \tau H_{NS}^{(3)}$ , which take values in the integer cohomology group  $H^3(X, \mathbb{Z})$ , also break the supersymmetry  $N = 2$  to  $N = 1$ . The  $\tau = C^{(0)} + ie^{-\varphi}$  is the complexified Type IIB coupling field. Its contribution to superpotentials is [5, 6]

$$\mathcal{W}_{flux}(z) = \int_X H_{RR}^{(3)} \wedge \Omega^{3,0} = \sum_\alpha N_\alpha \cdot \Pi^\alpha(z), \quad N_\alpha \in \mathbb{Z}. \quad (1.3)$$

The contributions of D-brane and background flux (here the NS-flux ignored) give together the general form of superpotential as follow [7, 8]

$$\mathcal{W}(z, \hat{z}) = \mathcal{W}_{brane}(z, \hat{z}) + \mathcal{W}_{flux}(z) = \sum_{\gamma_i \in H^3(Z^*, \mathcal{H})} N_i \Pi_i(z, \hat{z}) \quad (1.4)$$

where  $N_i = n_i + \tau m_\sigma$ ,  $\tau$  is the dilaton of type II string and  $\Pi_i$  is a relative periods defined in a relative cycle  $\Gamma \in H_3(X, D)$  whose boundary is wrapped by D-branes and  $D$  is a holomorphic divisor of the Calabi-Yau space. In fact, the two-cycles wrapped by the D-branes are holomorphic cycles only if the moduli are at the critical points of

the superpotentials. Thus, the two-cycles are generically not holomorphic. However, according to the arguments of [7–9], the non-holomorphic two-cycles can be replaced by a holomorphic divisor  $D$  of the ambient Calabi-Yau space with the divisor  $D$  encompassing the two-cycles.

Geometrically speaking, when varying the complex structure of Calabi-Yau space, a generic holomorphic curve will not be holomorphic with the respect to the new complex structure, and becomes obstructed to the deformation of the bulk moduli. The requirement for the holomorphy gives rise to a relation between the closed and open string moduli. Physically speaking, it turns out that the obstruction generates a superpotential for the effective theory depending on the closed and open string moduli.

The off-shell tension of D-branes,  $\mathcal{T}(z, \hat{z})$ , is equal to the relative period [7, 8, 10]

$$\Pi_{\Sigma} = \int_{\Gamma_{\Sigma}} \Omega(z, \hat{z}) \quad (1.5)$$

which measures the difference between the value of on-shell superpotentials for the two D-brane configurations

$$\mathcal{T}(z, \hat{z}) = \mathcal{W}(C^+) - \mathcal{W}(C^-) \quad (1.6)$$

with  $\partial\Gamma_{\Sigma} = C^+ - C^-$ . The domain wall tension is [11]

$$T(z) = \mathcal{T}(z, \hat{z})|_{\hat{z}=\text{critic points}} \quad (1.7)$$

where the critic points correspond to  $\frac{dW}{d\hat{z}} = 0$  [10] and the  $C^{\pm}$  is the holomorphic curves at those critical points. The critical points are alternatively defined as the Nother-Lefschetz locus [12]

$$\mathcal{N} = \{(z, \hat{z}) \mid \pi(z, \hat{z}; \partial\Gamma(z, \hat{z})) \equiv 0\} \quad (1.8)$$

where

$$\pi(z, \hat{z}; \partial\Gamma(z, \hat{z})) = \int_{\partial\Gamma} \omega_{\hat{a}}^{(2,0)}(z, \hat{z}), \quad \hat{a} = 1, \dots, \dim(H^{2,0}(D)) \quad (1.9)$$

and  $\omega_{\hat{a}}^{(2,0)}$  is an element of the cohomology group  $H^{(2,0)}(D)$ . At those critical points, the domain wall tensions are also known as normal function giving the Abel-Jacobi invariants [11–15]

The Superpotential can be calculated by study the Hodge variation on the cohomology group  $\Gamma \in H_3(X, D)$ . The flat Gauss-Manin connection on this cohomology

group can determine the mirror map between A-model and B-model. By the mirror symmetry, we can also obtain the Ooguri-Vafa invariants.

The purpose of this note is to calculate the off-shell superpotential which at the critical point equal to the domain wall tensions (on-shell superpotential) that have been obtained in the previous work [1].

## 2 Generalized GKZ system and Differential Operators

The period integrals can be written as

$$\Pi_i = \int_{\gamma_i} \frac{1}{P(a, X)} \prod_{j=1}^n \frac{dX_j}{X_j}. \quad (2.1)$$

where  $P$  is the hypersurface equation and  $a_i$  is the moduli determining the complex structure in B-model. See more in [1]. According to the refs. [16, 17], the period integrals can be annihilated by differential operators

$$\begin{aligned} \mathcal{L}(l) &= \prod_{l_i > 0} (\partial_{a_i})^{l_i} - \prod_{l_i < 0} (\partial_{a_i})^{l_i} \\ \mathcal{Z}_k &= \sum_{i=0}^{p-1} \nu_{i,k}^* \vartheta_i, \quad \mathcal{Z}_0 = \sum_{i=0}^{p-1} \vartheta_i - 1 \end{aligned} \quad (2.2)$$

where  $\vartheta_i = a_i \partial_{a_i}$ . As noted in refs. [18] [21], the equations  $\mathcal{Z}_k \Pi(a_i) = 0$  reflex the invariance under the torus action, defining torus invariant algebraic coordinates  $z_a$  on the moduli space of complex structure of  $X$  [11]:

$$z_a = (-1)^{l_0^a} \prod_i a_i^{l_i^a} \quad (2.3)$$

where  $l_a$ ,  $a = 1, \dots, h^{2,1}(X)$  is generators of the Mori cone, one can rewrite the differential operators  $\mathcal{L}(l)$  as [11, 17, 18]

$$\mathcal{L}(l) = \prod_{k=1}^{l_0} (\vartheta_0 - k) \prod_{l_i > 0} \prod_{k=0}^{l_i-1} (\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0} (\vartheta_0 - k) \prod_{l_i < 0} \prod_{k=0}^{l_i-1} (\vartheta_i - k). \quad (2.4)$$

The solution to the GKZ system can be written as [11, 17, 18]

$$B_{l^a}(z^a; \rho) = \sum_{n_1, \dots, n_N \in \mathbb{Z}_0^+} \frac{\Gamma(1 - \sum_a l_0^a (n_a + \rho_a))}{\prod_{i>0} \Gamma(1 + \sum_a l_i^a (n_a + \rho_a))} \prod_a z_a^{n_a + \rho_a}. \quad (2.5)$$

In this paper we consider the family of divisors  $\mathcal{D}$  with a single open deformation moduli  $\hat{z}$

$$x_1^{b_1} + \hat{z}x_2^{b_2} = 0 \quad (2.6)$$

where  $b_1, b_2$  are some appropriate integers. The relative 3-form  $\underline{\Omega} := (\Omega_X^{3,0}, 0)$  and the relative periods satisfy a set of differential equations [7–9, 11, 21]

$$\mathcal{L}_a(\theta, \hat{\theta})\underline{\Omega} = d\underline{\omega}^{(2,0)} \Rightarrow \mathcal{L}_a(\theta, \hat{\theta})\mathcal{T}(z, \hat{z}) = 0. \quad (2.7)$$

with some corresponding two-form  $\underline{\omega}^{(2,0)}$ . The differential operators  $\mathcal{L}_a(\theta, \hat{\theta})$  can be expressed as [11]

$$\mathcal{L}_a(\theta, \hat{\theta}) := \mathcal{L}_a^b - \mathcal{L}_a^{bd}\hat{\theta} \quad (2.8)$$

for  $\mathcal{L}_a^b$  acting only on bulk part from closed sector,  $\mathcal{L}_a^{bd}$  on boundary part from open-closed sector and  $\hat{\theta} = \hat{z}\partial_{\hat{z}}$ . The explicit form of these operators will be given in following model. From the (1.9) one can obtain

$$2\pi i \hat{\theta} \mathcal{T}(z, \hat{z}) = \pi(z, \hat{z}) \quad (2.9)$$

for only the family of divisors  $\mathcal{D}$  depending on the  $\hat{z}$ . Hence the off-shell superpotential can be obtained by integrating the period on subsystem  $\pi(z, \hat{z})$ .

### 3 Superpotentials of Hypersurface $X_{24}(1, 1, 2, 8, 12)$

The  $X_{24}(1, 1, 2, 8, 12)$  is defined as the zero locus of polynomial  $P$

$$P = x_1^{24} + x_2^{24} + x_3^{12} + x_4^3 + x_5^2 + \psi x_1 x_2 x_3 x_4 x_5 + \phi x_1^6 x_2^6 x_3^6 + \chi x_1^{12} x_2^{12} \quad (3.1)$$

The GLSM charge vectors  $l_a$  are the generators of the Mori cone as follows [18]

	0	1	2	3	4	5	6	7
$l_1$	−6	0	0	0	2	3	0	1
$l_2$	0	1	1	0	0	0	−2	0
$l_3$	0	0	0	1	0	0	1	−2.

(3.2)

The mirror manifolds can be constructed as an orbifold by the Greene-Plesser orbifold group acting as  $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$  with weights

$$\mathbb{Z}_6 : g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_6 : g_2 = (1, 0, -1, 0, 0), \quad \mathbb{Z}_3 : g_3 = (1, 0, 0, -1, 0) \quad (3.3)$$

where we denotes  $\lambda_{1,2}^6 = 1$  and  $\lambda_3^3 = 1$ .

By the generalized GKZ system, the period on the K3 surface has the form

$$\pi = \frac{c}{2} B_{\{\hat{l}_1, \hat{l}_2, \hat{l}_3\}}(u_1, u_2, u_3; \frac{1}{2}, \frac{1}{2}, 0) = -\frac{4c}{\pi^{\frac{3}{2}}} \sqrt{u_1 u_2 u_3} + \mathcal{O}((u_1 u_2)^{3/2}) \quad (3.4)$$

which vanishes at the critical locus  $u_2 = 0$ . According to (2.9), the off-shell superpotentials can be obtained by integrating the  $\pi$ :

$$\mathcal{T}_a^\pm(z_1, z_2, z_3) = \frac{1}{2\pi i} \int \pi(\hat{z}) \frac{d\hat{z}}{\hat{z}}, \quad (3.5)$$

with the appropriate integral constants [11], the superpotentials can be chosen as  $\mathcal{W}^+ = -\mathcal{W}^-$ . In this convention, the off-shell superpotentials can be obtained as

$$2\mathcal{W}^+ = \frac{1}{2\pi i} \int_{-\hat{z}}^{\hat{z}} \pi(\zeta) \frac{d\zeta}{\zeta}, \quad \mathcal{W}^\pm(z_1, z_2, z_3) = \mathcal{W}^\pm(z_1, z_2, z_3)|_{\hat{z}=1} \quad (3.6)$$

Eventually, The superpotential are

$$\mathcal{W}^\pm(z_1, z_2, z_3, \hat{z}) = \sum_{n_1, n_2, n_3} \frac{\mp c z_1^{\frac{1}{2}+n_1} z_2^{\frac{1}{2}+n_2} z_3^{n_3} \hat{z}^{\frac{-1-2n_2}{2}} \Gamma(6n_1+4)}{\Gamma(2+2n_2)\Gamma(2+2n_1)\Gamma(\frac{5}{2}+3n_1)\Gamma(1+n_3)\Gamma(n_3-2n_2)\Gamma(n_1-2n_3+\frac{3}{2})} \times \frac{\{(1-2n_2)_2 F_1(-\frac{1}{2}-n_2, -2n_2, \frac{1}{2}-n_2; \hat{z}) + \hat{z}(1+2n_2)_2 F_1((\frac{1}{2}-n_2, -2n_2, \frac{3}{2}-n_2; \hat{z}))\}}{4\pi(-1+4n_2^2)} \quad (3.7)$$

For calculation of instanton corrections, one need to know mirror map. The fundamental period  $\omega_0$  is solution of the Picard-Fuchs equation which we listed in [1]. The flat coordinates in A-model at large radius regime are related to the flat coordinates of B-model at large complex structure regime by mirror map  $t_i = \frac{\omega_i}{\omega_0}$ ,  $\omega_i := D_i^{(1)} \omega_0(z, \rho)|_{\rho=0}$ . The open-string mirror map are

$$\begin{aligned} q_1 &= z_1 + 312z_1^2 + 107604z_1^3 - z_1 z_3 - 192z_1^2 z_3 - z_1 z_3^2 + \mathcal{O}(z^4) \\ q_2 &= z_2 + 2z_2^2 + 5z_2^3 + z_2 z_4 + 3z_2^2 z_4 + z_2^2 z_4^2 + \mathcal{O}(z^4) \\ q_3 &= z_3 + 2z_3^2 + 3z_3^3 + 120z_3 z_1 + 41580z_1^2 z_3 + \mathcal{O}(z^4) \\ q_4 &= z_4 - z_4^2 + z_4^3 + \mathcal{O}(z^4) \end{aligned} \quad (3.8)$$

here  $q_i = e^{2\pi i t_i}$  and we can obtain the inverse mirror map

$$\begin{aligned} z_1 &= q_1 - 312q_1^2 + 87084q_1^3 + q_1 q_3 - 864q_1^2 q_3 + q_1 q_2 q_3 + \mathcal{O}(q^4) \\ z_2 &= q_2 - 2q_2^2 + 3q_2^3 + \mathcal{O}(q^4) \\ z_3 &= q_3 - 2q_3^2 + 3q_3^3 - 120q_1 q_3 + 10260q_1^2 q_3 + q_2 q_3 - 120q_1 q_2 q_3 + 600q_1 q_3^2 - 4q_2 q_3^2 + \mathcal{O}(q^4) \\ z_4 &= q_4 + q_4^2 + q_4^3 + \mathcal{O}(q^4) \end{aligned} \quad (3.9)$$

$d_4 = 0, d_3 = 1$ $d_1/2 \setminus d_2/2$	1	3	5	7	9
1	1	0	0	0	$\frac{-5}{2}$
3	-848	0	0	0	2120
5	-270978	0	0	0	677445
7	-4107040	0	0	0	10267600
9	-4859101222	0	0	0	12147753055

$d_4 = 0, d_3 = 2$ $d_1/2 \setminus d_2$	1	3	5	7	9
1	$\frac{-9}{16}$	$\frac{-9}{16}$	0	0	$\frac{45}{32}$
3	$\frac{521}{2}$	$\frac{521}{2}$	0	0	$\frac{-2605}{4}$
5	$\frac{-1397265}{8}$	$\frac{-2506065}{8}$	167400	-195120	$\frac{7890645}{16}$
7	100877911	205105111	-118540800	142047360	$\frac{-553418675}{2}$
9	$\frac{-160323502433}{8}$	$\frac{226729748767}{8}$	-64409331600	71920841760	$\frac{251804856805}{16}$

$d_4 = 1, d_3 = 1$ $d_1/2 \setminus d_2$	1	3	5	7	9
1	$\frac{-29}{18}$	0	0	$\frac{-7}{2}$	$\frac{-35}{36}$
3	$\frac{12296}{9}$	0	0	2968	$\frac{7420}{9}$
5	$\frac{1309727}{3}$	0	5130	943293	$\frac{1611485}{6}$
7	$\frac{59552080}{9}$	0	-3734640	18109280	$\frac{2324840}{9}$
9	$\frac{70456967719}{9}$	0	-1890907740	18897762017	$\frac{50997932065}{18}$

Table 1: Disc invariants  $n_{d_1, d_2, d_3, d_4}$  for the off-shell superpotential  $W_1$  of the 3-fold  $\mathbb{P}_{1,1,2,8,12}[24]$

Using the modified multi-cover formula [3] for this case

$$\frac{\mathcal{W}^\pm(z(q))}{w_0(z(q))} = \frac{1}{(2i\pi)^2} \sum_{k \text{ odd}} \sum_{d_3, d_4, d_1, 2 \text{ odd} \geq 0} n_{d_1, d_2, d_3, d_4}^\pm \frac{q_1^{kd_1/2} q_2^{kd_2/2} q_3^{kd_3} q_4^{kd_4}}{k^2}, \quad (3.10)$$

the superpotentials  $\mathcal{W}^+$  give Ooguri-Vafa invariants  $n_{d_1, d_2, d_3, d_4}$  for the normalization constants  $c = 1$ , which are listed in Table.1.

## 4 Superpotential of Hypersurface $X_{12}(1, 1, 1, 3, 6)$

The  $X_{12}(1, 1, 1, 3, 6)$  is defined as zero locus of  $P$ :

$$P = x_1^{12} + x_2^{12} + x_3^{12} + x_4^4 + x_5^2 + \psi x_1 x_2 x_3 x_4 x_5 + \phi x_1^4 x_2^4 x_3^4 \quad (4.1)$$

The GLSM charge vectors in this case are [18]

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l_1 & -4 & 0 & 0 & 0 & 1 & 2 & 1 \\ l_2 & 0 & 1 & 1 & 1 & 0 & 0 & -3 \end{array} \quad (4.2)$$

On the mirror manifolds, the Greene-Plesser orbifold group acts as  $x_i \rightarrow \lambda_k^{g_{k,i}} x_i$  with weights

$$\mathbb{Z}_6 : g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_4 : g_2 = (0, 1, 2, 1, 0) \quad (4.3)$$

where we denotes  $\lambda_1^6 = 1$ ,  $\lambda_2^4 = 1$ .

In [1], we have obtained the period in subsystem as follows

$$\pi(u_1, u_2) = \frac{c}{2} B_{\{\hat{l}_1, \hat{l}_2\}}(u_1, u_2; 0, \frac{1}{2}) \quad (4.4)$$

where  $c$  are some normalization constants not determined by the differential operator. According to (2.9), the off-shell superpotentials can be obtained by integrating the  $\pi$ :

$$\mathcal{T}_a^\pm(z_1, z_2, z_3) = \frac{1}{2\pi i} \int \pi(\hat{z}) \frac{d\hat{z}}{\hat{z}}, \quad (4.5)$$

with the appropriate integral constants [11], the superpotentials can be chosen as  $\mathcal{W}^+ = -\mathcal{W}^-$ .

Eventually, The superpotential are

$$\mathcal{W}^\pm(z_1, z_2, z_3, \hat{z}) = \sum_{n_1, n_2, n_3} \frac{\mp c z_1^{\frac{1}{2}+n_1} z_2^{n_2} \hat{z}^{\frac{-1-2n_1}{2}} \Gamma(4n_1 + \frac{5}{2})}{\Gamma(1+2n_2)\Gamma(1+n_2)\Gamma(\frac{3}{2}+n_1)\Gamma(2+2n_1)\Gamma(n_1-3n_2+\frac{3}{2})} \quad (4.6)$$

$$\frac{\{(1-2n_1)_2 F_1(-\frac{1}{2}-n_1, -2n_1, \frac{1}{2}-n_1; \hat{z}) + \hat{z}(1+2n_1)_2 F_1((\frac{1}{2}-n_1, -2n_1, \frac{3}{2}-n_1; \hat{z}))\}}{4\pi(-1+4n_1^2)}$$

For calculation of instanton corrections, one need to know mirror map. The fundamental period  $\omega_0$  is solution of the Picard-Fuchs equation which we listed in [1]. The flat



coordinates in A-model at large radius regime are related to the flat coordinates of B-model at large complex structure regime by mirror map  $t_i = \frac{\omega_i}{\omega_0}$ ,  $\omega_i := D_i^{(1)} \omega_0(z, \rho)|_{\rho=0}$ . The open-string mirror map are

$$\begin{aligned} q_1 &= z_1 + 40z_1^2 + 1876z_1^3 + 2z_1z_2 - 13z_1z_2^2 + z_1z_2z_3 + \mathcal{O}(z^4) \\ q_2 &= z_2 - 6z_2^2 + 63z_2^3 + z_2z_3 - 9z_2^2z_3 + \mathcal{O}(z^4) \\ q_3 &= z_3 - z_3^2 + z_3^3 + \mathcal{O}(z^4) \end{aligned} \quad (4.7)$$

here  $q_i = e^{2\pi i t_i}$  and we can obtain the inverse mirror map as follows

$$\begin{aligned} z_1 &= q_1 - 40q_1^2 + 1324q_1^3 - 2q_1q_2 + 268q_1^2q_2 + 5q_1q_2^2 + \mathcal{O}(q^4) \\ z_2 &= q_2 + 6q_2^2 + 9q_2^3 - 36q_1q_2 - 468q_1q_2^2 + 630q_1^2q_2 + \mathcal{O}(q^4) \\ z_3 &= q_3 + q_3^2 + q_3^3 + \mathcal{O}(q^4) \end{aligned} \quad (4.8)$$

$d_3 = 0$ $d_1/2 \setminus d_2$	0	1	2	3	4
1	1	$\frac{-13}{16}$	$\frac{2693}{1024}$	$\frac{19517}{9}$	$\frac{7703}{16384}$
3	$\frac{1312}{243}$	$\frac{68231}{1296}$	$\frac{-23305385}{82944}$	$\frac{-3519745}{18}$	$\frac{-1672979243}{3981312}$
5	$\frac{63544513}{28350}$	$\frac{135578197}{12960}$	$\frac{346285919719}{5806080}$	$\frac{-9330830923}{1944}$	$\frac{-1608130586479}{92897280}$
7	$\frac{172956753731}{1389150}$	$\frac{5372183267179}{3175200}$	$\frac{-21892937788889}{8128512}$	$\frac{32917422417037}{136080}$	$\frac{-282745996819463}{43352064}$
9	$\frac{13409490308809711}{600112800}$	$\frac{216480619417211431}{355622400}$	$\frac{-19644707820777819881}{13655900160}$	$\frac{-82475053081873279}{7620480}$	$\frac{311040357663729110033}{93640458240}$

$d_3 = 1$ $d_1/2 \setminus d_2$	0	1	2	3	4
1	$\frac{-10}{9}$	$\frac{175}{144}$	$\frac{-18923}{2304}$	$\frac{23077}{147456}$	$\frac{-132267135}{2097152}$
3	$\frac{3380}{81}$	$\frac{-82429}{432}$	$\frac{48484423}{41472}$	$\frac{1376117443}{1327104}$	$\frac{601661687053}{42467328}$
5	$\frac{-1138840}{567}$	$\frac{403544255}{18144}$	$\frac{-534921106991}{2903040}$	$\frac{166478391791}{3440640}$	$\frac{-20526886980289679}{5945425920}$
7	$\frac{3400299058}{42525}$	$\frac{-835235546479}{272160}$	$\frac{786036635335453}{60963840}$	$\frac{4714357006892647}{650280960}$	$\frac{-44180037787516945679}{62426972160}$
9	$\frac{-121337433752293}{33339600}$	$\frac{-99152104754391869}{152409600}$	$\frac{15374804216369862857}{8534937600}$	$\frac{-62693086142469434527}{6242697216}$	$\frac{-923658211082431780070641}{3995326218240}$

Table 2: Disc invariants  $n_{d_1, d_2, d_3}$  for the off-shell superpotential  $\mathcal{W}_1^+$  of the 3-fold  $\mathbb{P}_{1,1,1,3,6}[12]$

Using the modified multi-cover formula [3] for this case

$$\frac{\mathcal{W}^\pm(z(q))}{w_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k \text{ odd}} \sum_{d_1 \text{ odd}, d_2, d_3 \geq 0} n_{d_1, d_2, d_3}^\pm \frac{q_1^{kd_1/2} q_2^{kd_2} q_3^{kd_3}}{k^2}, \quad (4.9)$$

The superpotentials  $\mathcal{W}^+$  give Ooguri-Vafa invariants  $n_{d_1, d_2, d_3}$  for the normalization constants  $c = 1$ , which are listed in Table. 2

## 5 Summary

In this paper, we make a further step of previous work [1] and calculate the off-shell superpotential. By open mirror symmetry, we also compute the Ooguri-Vafa invariants from A-model expansion.

The superpotential of Type II string theory are important in both physics and mathematics. It also related to F-theory by open-closed duality [19–21]. In type II/F-theory compactification, the vacuum structure is determined by the superpotentials, whose second derivative gives the chiral ring structure. The quantum cohomology ring structure comes from the world-sheet instanton corrections and space-time instanton corrections [7, 8]. In fact, the more general vacuum structure of type II/F-theory/heterotic theory compactification can be tackled in Hodge variance approach.

In next work, we will study D-brane in general case. We also try to calculate the D-brane superpotential with the method of  $A_\infty$  structure of the derived category  $D_{\text{coh}}(X)$  and path algebras of quivers.

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## References

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